Lecture 4d: Dynamics of Collisionless Drift-waves and Trapped Electron Modes

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June 2021

In last two lectures, we talked about pinches, and two mechanisms of profile peaking: one is the thermo-electric theory where we used ion mixing mode as an example to show the free energy gradient drives the peaking; the other is the turbulent equi-partition pinch (TEP) where the geometric variation leads to the peaking due to the homogenization process.

In this lecture, we will move on to the discussion about collisionless drift-waves (DWs) and trapped electron modes (TEMs), which we will see that trapped electron modes can be treated as one application of collisionless drift-waves. We'll start from talking about the energetics in today's lecture, which is important for particle and heat transport problems.

1 Review – drift-waves and zonal flow

We have discussed the formulation of drift-waves in Lecture 3a-d. The Hasegawa-Mima (H-M) equation gives the basic description of drift-waves, and we discussed its evolution and transport properties in terms of PV (potential vorticity) conservation. Moreover, we extended the formulation by considering the parallel current to obtain the Hasegawa-Wakatani (H-W) equation, and realised that non-adiabatic electron response can lead to drift-wave instabilities.

One of the key features we discussed is the emergence of zonal flow from turbulence fluctuation, and the zonal flow (or zonal corrugation) can serve as a sink of fluctuation energy. The zonal flow are the radial components of $\langle\phi\rangle = \langle\phi\rangle(r)$, and the zonal corrugation, in simple drift-waves, is the zonal density δn. These zonal structures are key elements of mode spectrum with k_r only ($k_\theta, k_\parallel = 0$), and the waves refer to components with non-zero $k_\theta, k_\parallel$ and k_r. The main difference between these two is that the zonal structure does not induce radial flux, because $k_\theta = 0$ due to symmetry.

Zonal modes are modes with minimal consideration: (a). it has the minimal inertia, $-\rho_s^2 \frac{d}{dt}\nabla_\perp^2 \frac{e\phi}{T_e}$ as compared to $\frac{d}{dt}(1-\rho_s^2\nabla_\perp^2)\frac{e\phi}{T_e}$; (b). it induces zero transport, since $k_\theta = 0$ due to symmetry; (c). it's weakly damped ($k_\parallel = 0$), so that other modes can be easily coupled to the zonal structure.

After we discussed the drift-wave – zonal flow system, we now continue to the collisionless drift-wave and trapped electron modes. This system is a combination of resonant electrons (for collisionless DW, $\omega = k_\parallel v_{th}$; for TEM, $\omega = \omega_D \epsilon$) and the negative dissipation ($\omega - \omega_* < 0$). The resonant condition $\omega = \bar{\omega}_D \epsilon$ is also known as collisionless TEM (CTEM). The magnetic curvature drift becomes important as we bounce-average out the parallel streaming motion, so the wave frequency can resonate with the drift frequency $\omega = \bar{\omega}_D$. In principle, the resonant ions and ion Landau damping may also come into play, and a good example is the current-driven ion acoustic wave (CDIA). The current (mainly electron velocity) tends to destabilize, while the ion Landau damping regulates the state. The interplay between the ion acoustics and the current drive ($kC_s - kv_d$) is very similar to the negative dissapation ($\omega - \omega_*$) we discussed here. Moreover, the ion zonal structures, like ion flows, also lead to intrinsic rotation which will be discussed later in this class.

In the following two lectures, we will focus on two sections:

(a). The energetics in collisionless drift-waves. The energetics with resonant particles and with zonal structues are separately discussed in previous studies. For example, Sagdeev and Galeev, 1969 discussed the energetics with resonant particles in their classic lecture notes on *Nonlinear Plasma Theory*. Diamond et al., 2005 review discussed the energetics between waves and the zonal flow. In this lecture, we will put these together and discuss the energetics in general.

(b). Trapped electron modes. The TEM is more unstable compared to the collisionless drift-waves. The wave-particle interaction for collisionless drift-wave happens on the particle streaming time-scale, $1/\tau_T = |k_\parallel v_{th}|$, which can be extremely fast. TEM, on the other hand, has an interaction time-scale of $1/\tau_{Drift} = |\bar{\omega}_D|$. There's sufficient time to establish the wave-particle correlation in TEM, thus leading to a stronger mode. Also, TEM is universal, since it can be driven by density gradient ∇n and electron temperature gradient ∇T_e, meaning that TEM can still persist even when ITG becomes stable. There are more reasons for us to be interested in TEM, such as that trapped electrons can couple to ITG drive, and that TEM limits ∇n due to the induced particle transport. When considering α particles, these energetic particles tends to slow down and heat the electrons, which gives more free energy (∇P_e) to excite TEM.

2 Background knowledge on energetics

To study the energetics, we start from the quasi-linear theory in one-dimensional system (Q1D). The evolution of the average distribution function follows the mean-fluctuation equation,

$$\begin{aligned} \frac{\partial \langle f \rangle}{\partial t} &= \frac{\partial}{\partial v} D \frac{\partial}{\partial v} \partial \langle f \rangle \\ \text{where} \quad D &= \sum_k \frac{q^2}{m^2} |E_k|^2 \left[\pi \delta(\omega - kv) + \frac{|\gamma_k|}{\omega_k^2} \right] \end{aligned} \tag{1}$$

$\pi\delta(\omega - kv)$ is the response from the resonant particles, and $\frac{|\gamma_k|}{\omega_k^2}$ is the response from nonresonant particles.

To utilize the quasi-linear theory, there are two implications: Firstly, we need overlapped islands in phase space, which lead to stochasticity. This is the basis for the irreversibility of resonant diffusion. Secondly, to validate the use of unperturbed orbits, the scattering wave field must have a faster changing rate than the particle bouncing rate, $k\Delta(w/k) > 1/\tau_b$. That is, the wave is quickly changing before the particle streaming trajectories receive any influence.

There are two key energy theorems associated with the quasi-linear theory.

$$\partial_t(\text{TWED}) + \partial_t(\text{RPKED}) = 0 \tag{2}$$

$$\partial_t(\text{TEED}) + \partial_t(\text{PKED}) = 0 \tag{3}$$

where TWED - Total Wave Energy Density; RPKED - Resonant Particle Kinetic Energy Density; TEED - Total Electric Energy Density; PKED - Particle Kinetic Energy Density.

Both of them are telling the energy conservation of the system, and note that TWED = TEED + Nonresonant PKED. From the Poynting theorem, we can get the TWED evolution

$$\begin{aligned} &\partial_t(\text{TWED}) + \nabla \cdot (\text{WEDF}) + \langle \boldsymbol{E} \cdot \boldsymbol{J} \rangle = 0 \\ &\text{TWED} = \omega_k \left. \frac{\partial \epsilon}{\partial \omega} \right|_k \frac{|E_k|^2}{8\pi} \\ &\text{WEDF} = v_{gr} \left[\omega_k \left. \frac{\partial \epsilon}{\partial \omega} \right|_k \right] \frac{|E_k|^2}{8\pi} \end{aligned} \tag{4}$$

In the formula, $\langle \boldsymbol{E} \cdot \boldsymbol{J} \rangle$ is the heating term. WEDF refers to the wave energy density flux, which is simply TWED multiply group velocity $v_{gr} = \frac{d\omega}{dk}$. The group velocity can be very different for parallel and perpendicular directions. For example, in the drift-acoustic wave, the parallel group velocity is mainly from the acoustic coupling. If we look at the drift-acoustic dispersion relation,

$$1 + k_\perp^2 \rho_s^2 = \frac{\omega_*}{\omega} + \frac{k_\parallel^2 C_s^2}{\omega^2} \tag{5}$$

Apparently, if we omit the diamagnetic term, the parallel group velocity are just $v_{gr,\parallel} = \frac{\partial \omega}{\partial k_\parallel} = \frac{C_s}{1 + k_\perp^2 \rho_s^2}$, which stems from the parallel compressibility.

3 Energetics for drift-waves

Apply the energy theorem to the drift-waves,

$$\partial_t(\text{TWED}) + \partial_t(\text{REED}) + \partial_t(\text{RIED}) + \partial_t(\text{ZED}) = 0 \tag{6}$$

The total wave energy density is balanced with the resonant electron energy density (REED), resonant ion energy density (RIED and the zonal energy density (ZED). The resonent particle energies already appeared in previous section, $RPKED = REED + RIED$, while the zonal energy is newly added to our discussion.

In general, the resonant electrons drives the wave and input energy into the wave through electron cooling, the resonant ions damps the wave and extract the energy by ion heating. The zonal energy, which contains both the flow and corrugation, also extract energy from the wave in most cases. To have a complete energy balance, we should also have some knowledge on the zonal mode damping, such as viscosity, but it's independent from current total wave energy balance. Both fluid electrons (adiabatic Boltzmann electron) and ions support the waves and they all runs into the energy balance of total wave energy.

Now, we have three players in the energy balance: the resonant particles, wave modes, and zonal structures. The interaction between the three can be important for understanding energetic particles (EPs). An analog of zonal mode structure in one-dimensional system is soliton or double layer problem in space physics, namely, a large-scale structure emerged in the system.

Consider the zonal-mean conservation from the Hasegawa-Wakatani model:

$$\begin{aligned} \rho_s^2 \frac{d}{dt}\nabla_\perp^2 \frac{e\tilde{\phi}}{T_e} &= D_\parallel \nabla_\parallel^2 \left(\frac{e\tilde{\phi}}{T_e} - \frac{\tilde{n}}{n_0}\right) \\ \frac{1}{n_0}\frac{d}{dt}\tilde{n} + \tilde{v}_r \frac{1}{n_0}\frac{\partial\langle n\rangle}{\partial r} &= D_\parallel \nabla_\parallel^2 \left(\frac{e\tilde{\phi}}{T_e} - \frac{\tilde{n}}{n_0}\right) \end{aligned} \tag{7}$$

The zonal equation can then be obtained by averaging over θ and parallel direction, leaving radial dependency only,

$$\begin{aligned} \rho_s^2 \frac{\partial}{\partial t}\nabla_r^2 \frac{e\langle\phi\rangle}{T_e} + \frac{\partial}{\partial r}\left\langle \tilde{v}_r \nabla_r^2 \frac{e\tilde{\phi}}{T_e}\right\rangle &= 0 \\ \frac{\partial}{\partial t}\frac{\langle\delta n\rangle}{n_0} + \frac{\partial}{\partial r}\left\langle \tilde{v}_r \frac{\tilde{n}}{n_0}\right\rangle &= 0 \end{aligned} \tag{8}$$

where $\langle\delta n\rangle$ is the zonal corrugation and $\langle\phi\rangle$ is the zonal flow.

Although the zonal modes are coupled to the wave equations, they generally transfer in one-direction. That is, the zonal flow $\langle\phi\rangle$ is supported by the advection of wave perturbation $\tilde{\phi}$, and the density corrugation $\langle\delta n\rangle$ is supported by the radial advection of density perturbation $\tilde{n}$. The saturation of the zonal structures may thus require other damping mechanisms.

The total energy for the drift-wave can then be calculated as

$$\mathcal{E} = \int d^3x\, \rho C_s^2 \left[\left(\frac{\tilde{n}}{n_0}\right)^2 + \rho_s^2 \left(\nabla_\perp \frac{e\tilde{\phi}}{T_e}\right)^2\right] \tag{9}$$

The internal energy term $\left(\frac{\tilde{n}}{n_0}\right)^2$ contains both the wave energy and the corrugation energy, and the kinetic energy term $\left(\nabla_\perp \frac{e\tilde{\phi}}{T_e}\right)^2$ contains both the wave and the flow energy. This system energy density is analogous to the 1D ion-acoustic system discussed in physics 218a. More generally, the internal energy should contains all the pressure energy (density and temperature). The parallel kinetic energy should also be included in the internal energy if acoustic waves are considered in the formulation. For simplicity, in the later discussion we will neglect the normalization factors to write $\frac{e\tilde{\phi}}{T_e}$ as $\tilde{\phi}$ and $\frac{\tilde{n}}{n_0}$ as $\tilde{n}$.

Let's first consider the internal energy balance, from wave equation, eq.(7)

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}_{int,wave} &= \frac{\partial}{\partial t}\int d^3x\frac{\tilde{n}^2}{2} \\
&= -\int d^3x(\langle\tilde{n}\tilde{v}_r\rangle + \tilde{n}\tilde{v}_r)\frac{\partial}{\partial r}\left(\langle\delta n\rangle + \tilde{n}\right) + \int d^3x\tilde{n}D_{\parallel}\nabla_{\parallel}^2\left(\tilde{\phi} - \tilde{n}\right) \\
&= -\int d^3x\langle\tilde{n}\tilde{v}_r\rangle\frac{\partial}{\partial r}\langle\delta n\rangle - \int d^3x\langle\tilde{n}\tilde{v}_r\rangle\frac{\partial}{\partial r}\tilde{n} - \int d^3x\ \tilde{n}\tilde{v}_r\frac{\partial}{\partial r}\langle\delta n\rangle \\
&\quad - \int d^3x\ \tilde{n}\tilde{v}_r\frac{\partial}{\partial r}\tilde{n} + \int d^3x\tilde{n}D_{\parallel}\nabla_{\parallel}^2\tilde{\phi} - \int d^3x\tilde{n}D_{\parallel}\nabla_{\parallel}^2\tilde{n}
\end{aligned} \tag{10}
$$

If we notice that the symmetry requires $\int d\theta dz\ \tilde{n} = \int d\theta dz\ \tilde{n}\tilde{v}_r = 0$, we instantly remove $\int d^3x\langle\tilde{n}\tilde{v}_r\rangle\frac{\partial}{\partial r}\tilde{n} = \int d^3x\ \tilde{n}\tilde{v}_r\frac{\partial}{\partial r}\langle\delta n\rangle = 0$. Then, do the integral by parts, we can rewrite $D_{\parallel}$ terms and get

$$
\frac{\partial}{\partial t}\mathcal{E}_{int,wave} = -\int d^3x\langle\tilde{n}\tilde{v}_r\rangle\frac{\partial}{\partial r}\langle\delta n\rangle - \int d^3x\ \tilde{n}\tilde{v}_r\frac{\partial}{\partial r}\tilde{n} - \int d^3x\nabla\tilde{n}D_{\parallel}\nabla_{\parallel}\tilde{\phi} + \int d^3xD_{\parallel}(\nabla_{\parallel}\tilde{n})^2 \tag{11}
$$

The first term denotes the energy flow into the zonal density corrugation. If we do a similar calculation for the zonal corrugation, eq.(8),

$$
\frac{\partial}{\partial t}\mathcal{E}_{int,zonal} = \frac{\partial}{\partial t}\int d^3x\frac{\langle\delta n\rangle^2}{2} = -\int d^3x\langle\delta n\rangle\frac{\partial}{\partial r}\langle\tilde{v}_r\tilde{n}\rangle = \int d^3x\ \langle\tilde{v}_r\tilde{n}\rangle\ \frac{\partial}{\partial r}\langle\delta n\rangle \tag{12}
$$

We can see that the energy is balanced between the wave and the zonal structure. The total internal energy now gives

$$
\frac{\partial}{\partial t}\mathcal{E}_{int} = -\int d^3x\ \tilde{n}\tilde{v}_r\frac{\partial}{\partial r}\tilde{n} + \int d^3xD_{\parallel}\nabla_{\parallel}\tilde{n}(\nabla_{\parallel}\tilde{n} - \nabla_{\parallel}\tilde{\phi}) \tag{13}
$$

where the first term is the wave-wave coupling. The second term relates to the parallel collisionality $D_{\parallel}$.

For the kinetic energy, again we can obtain the energy balance for the wave and the zonal flow,

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}_{k,wave} &= \rho_s^2\int d^3x\frac{\partial}{\partial t}\frac{(\nabla_r\tilde{\phi})^2}{2} = -\int d^3x\ \tilde{\phi}\rho_s^2\frac{\partial}{\partial t}\nabla_r^2\tilde{\phi} \\
&= \int d^3x\langle\tilde{\phi}\tilde{v}_r\rangle\frac{\partial}{\partial r}(\rho_s^2\nabla_r^2\langle\phi\rangle) + \int d^3x\ \tilde{\phi}\tilde{v}_r\frac{\partial}{\partial r}(\rho_s^2\nabla_r^2\tilde{\phi}) \\
&\quad + \int d^3xD_{\parallel}(\nabla_{\parallel}\tilde{\phi})^2 - \int d^3x\nabla\tilde{\phi}D_{\parallel}\nabla_{\parallel}\tilde{n} \\
&= -\rho_s^2\int d^3x\langle\tilde{v}_r\nabla_r\tilde{\phi}\rangle\frac{\partial}{\partial r}\nabla_r\langle\phi\rangle - \rho_s^2\int d^3x\ \tilde{v}_r\nabla_r\tilde{\phi}\frac{\partial}{\partial r}\nabla_r\tilde{\phi} \\
&\quad + \int d^3xD_{\parallel}\nabla_{\parallel}\tilde{\phi}(\nabla_{\parallel}\tilde{\phi} - \nabla_{\parallel}\tilde{n})
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}_{k,zonal} &= \rho_s^2\int d^3x\frac{\partial}{\partial t}\frac{(\nabla_r\langle\phi\rangle)^2}{2} = -\int d^3x\langle\phi\rangle\rho_s^2\frac{\partial}{\partial t}\nabla_r^2\langle\phi\rangle \\
&= \rho_s^2\int d^3x\langle\phi\rangle\frac{\partial}{\partial r}\left\langle\tilde{v}_r\nabla_r^2\tilde{\phi}\right\rangle = \rho_s^2\int d^3x\left\langle\tilde{v}_r\nabla_r\tilde{\phi}\right\rangle\frac{\partial}{\partial r}\nabla_r\langle\phi\rangle
\end{aligned} \tag{15}
$$

The Taylor idensity is used to derive eq.(15), i.e. $\left\langle\tilde{v}_r\nabla_r^2\tilde{\phi}\right\rangle = \nabla_r\left\langle\tilde{v}_r\nabla_r\tilde{\phi}\right\rangle$. Similarly, the zonal flow is balanced with the wave kinetic energy, and only the wave-wave coupling terms and parallel collisionality terms remains. For Boltzmann response $\tilde{\phi} = \tilde{n}$, so that the collisional terms cancel each other if we sum over the internal and the kinetic energy. $D_{\parallel}$ terms should be a higher-order residue for the energy conservation.

Back to the interaction between the zonal corrugation and the drift-wave, the energy transport is realized by the radial flux $\langle\tilde{v}_r\tilde{n}\rangle$. For H-W system, i.e. electron drift-wave, the zonal density effect can be weak for

$\alpha = \frac{k_\parallel^2 v_{th}^2}{\omega \nu_{coll}} > 1$, and the zonal flow should dominate the case. To show this, we've already calculated the flux due to EDW in Lecture 3b,

$$\langle \tilde{v}_r \tilde{n} \rangle = \langle \tilde{v}_r \tilde{h} \rangle = \sum_k \rho_s C_s \left| \frac{e\tilde{\phi}}{T_e} \right|^2 \frac{k_\theta (k_\perp^2 \rho_s^2) \omega_*}{(1 + k_\perp^2 \rho_s^2) k_\parallel^2 D_\parallel} \tag{16}$$

which is on the order of $\mathcal{O}(\frac{\omega}{k_\parallel^2 D_\parallel}) = \mathcal{O}(\frac{1}{\alpha})$. However, for collisionless TEM, this isn't true as we will see in later discussion. For CTEM,

$$\langle \tilde{v}_r \tilde{n} \rangle \cong -\sum_k \langle \tilde{v}_r \rangle_k^2 \frac{\Delta \omega_k}{\omega_*^2} \nabla \langle n \rangle - D_{res} \nabla \langle n \rangle \tag{17}$$

The non-resonant response is a nonlinear wave-particle processes, leading to a beat-wave interaction $\pi\delta(\omega + \omega_* - \bar{\omega}_D \epsilon - \bar{\omega}'_D \epsilon)$ and nonlinear Landau damping of trapped electrons. $\Delta\omega_k$ is on the order of $(\frac{e\phi}{T_e})^2$, which results in a total of $(\frac{e\phi}{T_e})^4$ dependency for the non-resonant part. The resonant contribution D_{res} then plays an important role to give interesting coupling of resonant electrons to the density corrugation.

4 Energetics for collisionless drift-waves

For collisionless drift-waves, recall the basic assumption, $v_{thi} \leq \frac{\omega}{k_\parallel} \ll v_{the}$, that the phase velocity is between the ion thermal velocity and the electron thermal velocity. By including the non-adiabatic response, $\frac{\tilde{n}_e}{n} = \frac{e\tilde{\phi}}{T_e} + \tilde{h}$, the corrected Hasegawa-Mima model ("$i\delta$" model) with the zonal flow turns into

$$\begin{aligned} &\frac{\partial}{\partial t}\tilde{\phi} + \frac{\partial}{\partial t}\tilde{h} - \rho_s^2 \frac{d}{dt}\nabla_\perp^2 \tilde{\phi} + v_* \frac{\partial \tilde{\phi}}{\partial y} = 0 \\ &\rho_s^2 \frac{d}{dt}\nabla_\perp^2 \tilde{\phi} = \rho_s^2 \frac{\partial}{\partial t}\nabla_\perp^2 \tilde{\phi} + \rho_s^2 \frac{\partial}{\partial r}\left\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \right\rangle = 0 \end{aligned} \tag{18}$$

Here, the non-adiabatic part does not contribute to the zonal flow, and again the normalization factors are neglected $\frac{e\tilde{\phi}}{T_e} \to \tilde{\phi}$. If we use the approximation $\tilde{n} = \tilde{\phi}$, the energetics of the internal wave energy then gives

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{E}_{int,wave} &= \frac{\partial}{\partial t}\int d^3x \frac{\tilde{\phi}^2}{2} \\ &= -\int d^3x \left\langle \tilde{\phi}\frac{\partial}{\partial t}\tilde{h} \right\rangle + \int d^3x \; \tilde{\phi}\frac{d}{dt}(\rho_s^2 \nabla_r^2 \langle \phi \rangle) \\ &= -\int d^3x \left\langle \tilde{\phi}\frac{\partial}{\partial t}\tilde{h} \right\rangle - \rho_s^2 \int d^3x \left\langle \tilde{v}_r \nabla_r \tilde{\phi} \right\rangle \frac{\partial}{\partial r}\nabla_r \langle \phi \rangle \end{aligned} \tag{19}$$

The first term is the resonant electron coupling, that the non-adiabatic response results in electric work $\tilde{h}\phi$ with appropriate phase shift.

The second term is the coupling to zonal corrugation (for $\tilde{n} = \tilde{\phi}$, zonal flow and density corrugation are following the same equation), which can be calculated through energy balance, $\frac{\partial}{\partial t}\mathcal{E}_{wave,flow} = -\frac{\partial}{\partial t}\mathcal{E}_{zonal}$, and the calculation for zonal internal energy is the same as for eq.(15). If we notice that $v_y = -\nabla_r \tilde{\phi}$, then

$$\frac{\partial}{\partial t}\mathcal{E}_{zonal} = \rho_s^2 \int d^3x \frac{\partial}{\partial t}\frac{(\nabla_r \langle \phi \rangle)^2}{2} = \rho_s^2 \int d^3x \left\langle \tilde{v}_r \nabla_r \tilde{\phi} \right\rangle \frac{\partial}{\partial r}\nabla_r \langle \phi \rangle = \rho_s^2 \int d^3x \left\langle \tilde{v}_r \tilde{v}_y \right\rangle \langle v_y \rangle' \tag{20}$$

where the angular velocity, $\langle v_y \rangle' = \nabla_r(-\nabla_r \langle \phi \rangle) = \nabla_r v_y$, is just the radial shear of $E \times B$ (circular) velocity from the zonal flow. The energy input to the zonal flow is exactly the Reynolds' power generated by the Reynolds stress term.

In summary, the energy balance for the internal wave energy is now clear: one is the resonant electron drive interaction; the other is the Reynolds power exerted on zonal mode. If we want to further include ion Landau damping, we need to include the parallel compression, $\nabla \cdot \boldsymbol{v} \to \nabla_\perp \cdot v_\perp + \nabla_\parallel \cdot v_\parallel$. The extra contribution, $-\int d^3x \langle \tilde{\phi} \nabla_\parallel \tilde{v}_\parallel \rangle$, in eq.(19) needs to be calculated through ion drift-kinetic (or gyro-kinetic) equation.

To calculate the non-adiabatic response, we start from the linearized electron drift-kinetic equation,

$$\frac{\partial}{\partial t} \tilde{f} + v_\parallel \nabla_\parallel \tilde{f} - \frac{e}{B} \nabla \phi \times \hat{\boldsymbol{z}} \cdot \nabla f_0 - \frac{e}{m_e} E_\parallel \frac{\partial}{\partial v_\parallel} f_0 = 0 \tag{21}$$

Then, the perturbed distribution function is obtained,

$$\begin{aligned} \tilde{f}_k &= \frac{\frac{c}{B} i k_\theta \phi_k \frac{\partial f_0}{\partial r} + \frac{e}{m_e}(-ik_\parallel)\phi_k \frac{\partial f_0}{\partial v_\parallel}}{-i(\omega - k_\parallel v_\parallel)} \\ &= \frac{\frac{c}{B} i k_\theta \phi_k \frac{1}{n_0}\frac{\partial n_0}{\partial r} f_0 + \frac{e}{m_e} \frac{i(\omega - k_\parallel v_\parallel) - i\omega}{v_\parallel} \phi_k \frac{\partial f_0}{\partial v_\parallel}}{-i(\omega - k_\parallel v_\parallel)} \\ &= \frac{-i\omega_* - [i(\omega - k_\parallel v_\parallel) - i\omega]}{-i(\omega - k_\parallel v_\parallel)} \frac{e\phi_k}{T_e} f_0 \\ &= \left[1 - \frac{\omega - \omega_*}{\omega - k_\parallel v_\parallel}\right] \frac{e\phi_k}{T_e} f_0 \end{aligned} \tag{22}$$

Here, we assumed that the electrons are following a Maxwellian distribution, $f_0 = n_0 \left(\frac{m_e}{2\pi T_e}\right)^{3/2} \exp\left[-\frac{m_e v^2}{2T_e}\right]$, and the diamagnetic frequency is defined as $\omega_* = -\frac{1}{n_0}\frac{\partial n_0}{\partial r}\frac{cT_e}{eB} k_\theta$. Note that $\frac{\omega}{k_\parallel v_{th}} \ll 1$, the non-adiabatic response of resonant electrons is then calculated as $\tilde{h} = \int d^3v \tilde{f} - \frac{e\tilde{\phi}}{T_e}$.

$$\tilde{h}_k = -\frac{\omega - \omega_*}{k_\parallel v_{th}} \frac{e\phi_k}{T_e} \int d^3v \, \frac{v_{th}}{\omega/k_\parallel - v_\parallel} f_0 = i\pi \frac{\omega - \omega_*}{k_\parallel v_{th}} \frac{e\phi_k}{T_e} [v_{th} f_0]|_{v_\parallel = \frac{\omega}{k_\parallel}} \tag{23}$$

The linear dispersion relation follows,

$$1 + k_\perp^2 \rho_s^2 + i\pi \frac{\omega - \omega_*}{k_\parallel v_{th}} [v_{th} f_0]|_{v_\parallel = \frac{\omega}{k_\parallel}} - \frac{\omega_*}{\omega} = 0 \tag{24}$$

Since the parallel streaming is much faster than the phase frequency, we can expect $\tilde{h}$ to be weakly coupled. Thus have the linear instability

$$\omega_r = \frac{\omega_*}{1 + k_\perp^2 \rho_s^2}, \quad i\omega_i = -i\pi \frac{\omega_r - \omega_*}{k_\parallel v_{th}(1 + k_\perp^2 \rho_s^2)} [v_{th} f_0]|_{v_\parallel} \tag{25}$$

The growth rate is thus on the order of $\mathcal{O}(\frac{\omega}{k_\parallel v_{th}})$, which is marginally unstable.

Now, we can move on to the quasi-linear theory,

$$\frac{\partial}{\partial t} \tilde{f} + \frac{\partial}{\partial r} \left\langle \tilde{v}_r \tilde{f} \right\rangle + \frac{\partial}{\partial v_\parallel} \left\langle -\frac{e}{m_e} E_\parallel \tilde{f} \right\rangle = 0 \tag{26}$$

Multiply $\frac{v^2}{2}$ and integrate over velocity space, we can get the energy conservation,

$$\frac{\partial}{\partial t} \mathcal{E}_p + \frac{\partial}{\partial r} Q_r - \left\langle E_\parallel J_\parallel \right\rangle = 0 \tag{27}$$

The kinetic energy of electrons $\mathcal{E}_p$ is balanced with the turbulence heat flux Q_r and the Ohmic heating $\left\langle E_\parallel J_\parallel \right\rangle$. This energy equation can be compared with the wave energy theorem, (people who are not familiar with this may refer to Landau and Lifshitz, *Electrodynamics of continuous media*)

$$\frac{\partial}{\partial t} \mathcal{E}_w + \frac{\partial}{\partial r} S_r + \langle \boldsymbol{E} \cdot \boldsymbol{J} \rangle = 0 \tag{28}$$

the wave energy $\mathcal{E}_w$ is determined by the energy flux through boundary S_r and the electric power $\langle \boldsymbol{E} \cdot \boldsymbol{J} \rangle$. For details, the electron cooling provides the input of parallel eletric power, while the perpendicular power is balanced with the Reynolds power,

$$\langle \boldsymbol{E}_\perp \cdot \boldsymbol{J}_\perp \rangle = \langle -\nabla_\perp \phi \cdot \boldsymbol{J}_\perp \rangle = \langle \phi \nabla_\perp \cdot \boldsymbol{J}_\perp \rangle = \left\langle -q n_0 \rho_s^2 \phi \frac{d}{dt} \nabla_\perp^2 \phi \right\rangle \tag{29}$$

We can see that this perpendicular energy is exactly the Reynolds power calculated in eq.(19).